## The problem for the day

This first day of the IDSM Institute is devoted to an analysis of a single problem. We will find that many important mathematical ideas emerge when we look carefully at this problem, generalize it and extend it, and try to understand it at a deeper level. Our suggestion is to first spend some time solving the problem given here then move to the guided tour.

The problem that we will analyze is based on a problem from a high school pre-calculus text*:
Of all lines through the point $(5,2)$, find the line that cuts off the triangle of smallest area in the first quadrant.


Figure 1: $\quad$ What line through $(5,2)$ cuts off the smallest area?
*The original problem idea came from Advanced Mathematics: A Pre-calculus Course, (Revised Edition,1984), by Richard G. Brown and David P. Robbins, Houghton Mifflin, p. 19.

## Analyzing the problem: a guided tour

There are 12 questions below that represent a "guided tour" through an extended analysis of this problem. Participants need to go through at least the first 8 of these questions to gain an appreciation of what it means to analyze this problem in depth.

Getting a feel for the problem
By making rough sketches, get a feel for how the area of the triangle in Figure 1 varies as different lines are chosen.

1a) Look at extreme cases.
1b) Calculate areas for some easy intermediate cases.

In the questions that follow, $\$ 2$ to $\S 5$ represent an algebraic approach and $\S 6$ to $\S 9$ represent a geometric approach. See the flow chart on page 2. Either approach can serve as a starting point. But it is important not to neglect the geometric approach.

## Algebraic approach

## §2 Parameterizing the lines

Since the problem asks for a particular line, we have to have a way of parameterizing different lines that pass through the point (5,2). (To "parameterize" a line means stating one or more numbers, such as slope and an intercept, that identify the line uniquely.)
2a) Find as many different parameters as you can that will identify a unique line through (5,2).
2b) Give some explicit relationships among the parameters.
2c) In general, how many parameters does it take to identify a line through $(5,2)$ ?

## §3

## Representing the area

The area of a triangle formed by a line through $(5,2)$ can be expressed as a function of any parameter that identifies the line.

3a) Represent the area in terms of each of the parameters from $\$ 2$.
3b) Find a parameter that gives a particularly simple representation of area.
3c) Find a parameter that gives a particularly awkward representation of area.

## §4 Finding the minimum area

Once the area has been represented as a function of one variable, the problem reduces to finding the minimum of the function.

4a) Graph a few of the area functions from $\S 3$, and estimate the minimum area.
4b) Find the minimum using calculus methods.
4c) Of all the methods for finding the minimum, does any stand out as being simpler or more elegant?
(In $\$ 10$ below, an additional method of finding the minimum is explored using an inequality between the arithmetic mean and the geometric mean of two numbers.)

## §5 Generalizing the initial result

The work done in $\S 4$ has solved the original problem involving the specific point $(5,2)$. Try now to solve the problem for an arbitrary point ( $p, q$ ) in the first quadrant:

Of all lines through the point ( $p, q$ ), find the line that cuts off the triangle of smallest area in the first quadrant.
5a) Look carefully at the solution to the original problem and characterize the relationship the area-minimizing line has to the point $(5,2)$.
5b) Make a conjecture about the general case.
5c) Find the answer for the general case using calculus.
(In §9 below, there is a question based on the case where the lines don't meet at a right angle.)

## Geometric approach

## §6 Seeing this as a purely geometric problem

The original problem (Figure 1) is stated using the terminology of a coordinate system. One way to think about it differently is to remove the co-ordinate system altogether. This leaves a problem stated in purely geometric terms:

Given an angle ABC and a point $P$ inside the angle, what line through $P$ cuts off the triangle of least area?

The following questions represent an approach to solving this new problem.
6a) Make a conjecture about a geometric property that characterizes the line through a fixed point that cuts off the triangle of minimum area. (You may want to look again at your work in the previous sections.)
6b) Give a geometric proof of this conjecture based on the following approach: Add a line through $P$ to Figure 2 that is assumed to have the property in (6a), then add any other line through P . Now prove that this other line will cut off a larger area.

## §7 Special lines

Intuitively, most people realize that the line that cuts off the smallest area is not arbitrary, but is "special" in some sense. The problem posed in §6 and its solution involve what we can call geometrically "special" lines. They are lines characterized by a simple geometric property.
7a) The problem posed in $\S 6$ is a minimization problem. Pose a few other minimization problems, based on Figure 2, that involve minimization of some property other than area. (Don't try to solve these problems here. Some are difficult, but posing them is easy.)
$7 b)$ The geometric property in (6a) characterizes a special line through $P$ that is a solution to the area minimization problem. What other geometric properties can you think of that characterize special lines through P? (Don't try here to think of what minimization problem they might solve. This can be hard, but thinking up properties is easier.)

## Behold!

Some journals have sections called "Proof without words" or simply "Behold!". These sections often attempt to make a result transparent by giving a carefully chosen diagram. Show how the following diagram can be used in such a way that makes the result obtained in $\S 4$ and $\S 5$ above "obvious" at a glance. This also supports other solutions like in $\mathbf{6 b}$.


## Other ways to extend the analysis

§9 Generalization to angles other than right angles


9a) Formulate the problem for a case where the axes are not perpendicular.
9b) Solve the problem in this case by adapting the method used in $\S 5$.
§10 Using the "arithmetic mean - geometric mean" inequality.
The "arithmetic mean - geometric mean" inequality theorem states that for positive numbers $p$ and $q$, we have $\sqrt{p q} \leq \frac{p+q}{2}$, with equality occurring if and only if $p=q$. This theorem provides a method for finding maxima or minima in certain situations.

10a) Find the minimum area using the arithmetic mean - geometric mean inequality
10b) Prove this inequality.
§11 Further extension of the result
We have extended the original problem (Figure 1) to a geometric problem involving a point in an arbitrary angle §6, Figure 2). Actually the same "midpoint" property of the line that minimizes area holds for even more general regions such as those where a line is drawn through a point inside a circle or a parabola.
11a) Illustrate this midpoint property for a circle and for a parabola.
11b) Sketch an informal proof that the midpoint property characterizes the areaminimizing line for a point inside an arbitrary convex region.
§12 Summary
What are the major lessons learned from this analysis?

